actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

2717. Proposed by ENOS W. WITMER, Sophomore in Franklin and Marshall College.

Determine the integral values of m and n for which the equation $x^4 + mx^2y^2 + ny^4 = z^2$ has non-trivial solutions. [Carmichael's *Diophantine Analysis*, Prob. 13, p. 53.]

The following problems in volumes XX to XXIII still remain unsolved:
Algebra: numbers 406, 430, 461;
Geometry: numbers 446, 470, 472, 478, 494, 501, 510, 519, 523;
Calculus: numbers 339, 340, 342, 348, 353, 360, 385, 411, 415, 425, 429, 432, 434, 436, 440;
Mechanics: numbers 277, 279, 285, 287, 291, 308, 309, 313, 315, 322, 328, 335, 343, 344, 348, 350, 351, 356, 357;
Number Theory: numbers 189, 190, 191, 192, 200, 205, 231, 232, 234, 239, 245, 247, 260, 261, 263, 270, 271, 273, 274, 275.
The editors will be glad to receive solutions of any of these unsolved problems.

SOLUTIONS OF PROBLEMS.

Note. 1. Florence P. Lewis sent in a solution of 486 and Albert Babbitt a solution of 493, Algebra, after selection for publication had been made and sent to the Editor-in-Chief.

2. The following correction should be made: On page 24, of the January, 1918, number of the MONTHLY, 16th line from bottom, for $(n^2 + 1) - 4n$, read $(n + 1)^2 - 4n$. EDITORS.

427 (Calculus). Proposed by ROGER S. JOHNSON, Adelbert College, Cleveland, O.

Of all ellipses cirumscribed about a given parallelogram, the minimum (maximum) with regard to area has as conjugate diameters the diagonals of the parallelogram.

II. SOLUTION BY O. D. KELLOGG, University of Missouri.

I venture to add my solution to those given in the January number of the MONTHLY because it illustrates the fruitfulness of the notion "shear," a simple transformation which should doubtless find more use in elementary mathematics.[1] The coördinate axes having any position in the plane, the transformation $x = x'$, $y = y' - ax'$ defines a shear. The following are invariants: area, ellipse, conjugacy, parallelism.

Suppose E' and E'' are two ellipses circumscribed about the parallelogram P, the former having the diagonals of P as conjugate diameters. A shear may be found which carries E' over into a circle C', and consequently P into a square R', since its diagonals are conjugate diameters of a circle. Using the letters to denote areas, we have $P/E' = R'/C'$.

A second shear may be found which will carry E'' into a circle C'', and consequently P into a rectangle R'', and we have $P/E'' = R''/C''$.

But a square has an area whose ratio to that of the circumscribed circle is greater than that of any other rectangle. Hence $R'/C' > R''/C''$, and the above equations yield $E'' > E'$, so that E' is the circumscribed ellipse of *minimum* area.

438 (Calculus). Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the locus of the equation

$$y^6 - 3(a^2 - x^2)y^4 - 2ax^2y^3 + 3(a^2 - x^2)^2y^2 - 6ax^2(a^2 - x^2)y + a^2x^4 - (a^2 - x^2)^3 = 0,$$

first showing that it can be reduced to the form

$$y = kx^n \pm (a^2 - x^2)^m,$$

and finding the points of maximum abscissa, of maximum ordinate, and of inflection.

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[1] In fact the idea is used in Young and Morgan's *Elementary Analysis*, pages 132, 288, and 289.

SOLUTION BY THE PROPOSER.

Inspection of the two equations shows that $m = \frac{1}{2}$, $n = \frac{2}{3}$, and since the first equation is homogeneous in (a, x, y), $k = a^{1/3}$. The given equation is in fact the result of rationalizing

$$y = a^{1/3}x^{2/3} \pm \sqrt{a^2 - x^2}.$$

Let $x/a = \sin\theta$. Then $y/a = \sin^{2/3}\theta \pm \cos\theta$. Hence, $y'/a = 2/3 \sin^{-1/3}\theta \mp \tan\theta$, since $d\theta/dx = \sec\theta$. Hence, $y''/a = -2/9 \sin^{-4/3}\theta \mp \sec^3\theta$. When $y' = 0$, $\tan\theta \cdot \sin^{1/3}\theta = 2/3$. Solving, $\theta_1 = 38° \ 4.2'$, $x_1/a = 0.6166$, and $y_1/a = .7245 \pm .7873 = 1.5117$ or -0.0628. The first value gives a maximum ordinate. When $y'' = 0$, $\tan^{.8}\theta \sec\theta = (2/9)^{.6}$. Solving, $\theta_2 = 17° \ 1'$, $x_2/a = 0.292\bar{7}$, and $y_2/a = .4408 \pm .9562 = 1.3970$ or -0.5154. $y_2/a' = 1.004\bar{1} \mp .306\bar{1} = 0.6981$ or 1.3102.

The second value gives an inflection.

The curve is readily constructed by adding ordinates of the semi-cubical parabola and the circle.

(a, a) is evidently the point of maximum abscissa.

Also solved by ADELE HOLTWICK.

439 (Calculus). Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

SOLUTION BY C. C. YEN, Tangshan, North China.

Let $P \equiv (x, y, z)$ be the vertex of the rectangular parallelopiped lying in the first octant of the ellipsoid. Then the volume of the parallelopiped is $V = 8xyz$.

Since P lies on the ellipsoid, the coördinates (x, y, z) satisfy the equation of the ellipsoid, and therefore

$$V = 8xyz = 8cxy(1 - x^2/a^2 - y^2/b^2)^{1/2} = 8c \cdot F(x, y), \tag{1}$$

where $F(x, y) \equiv x \cdot y(1 - x^2/a^2 - y^2/b^2)^{1/2}$ is maximum when and only when V is maximum.

Differentiating, we get

$$\begin{aligned} \frac{\partial F}{\partial x} &= y(1 - 2x^2/a^2 - y^2/b^2) \div (1 - x^2/a^2 - y^2/b^2)^{1/2}, \\ \frac{\partial F}{\partial y} &= x(1 - x^2/a^2 - 2y^2/b^2) \div (1 - x^2/a^2 - y^2/b^2)^{1/2}. \end{aligned} \tag{2}$$

Equating to zero the left-hand members of (2), we have

$$2b^2x^2 + a^2y^2 = a^2b^2, \qquad b^2x^2 + 2a^2y^2 = a^2b^2,$$

which give

$$x^2 = \frac{a^2}{3}, \quad y^2 = \frac{b^2}{3}; \quad \text{and, therefore,} \quad x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}.$$

Differentiating (2), we get

$$\frac{\partial^2 F}{\partial x^2} = -\frac{xy}{a^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-3/2}\left(3 - \frac{2x^2}{a^2} - \frac{3y^2}{b^2}\right), \quad = -\frac{4b}{a\sqrt{3}} \text{ when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}};$$

$$\frac{\partial^2 F}{\partial y^2} = -\frac{xy}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-3/2}\left(3 - \frac{3x^2}{a^2} - \frac{2y^2}{b^2}\right), \quad = -\frac{4a}{b\sqrt{3}} \text{ when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}};$$

and

$$\frac{\partial^2 F}{\partial y \partial x} = \left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right)\left\{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} + \frac{y^2}{b^2}\right\} - \frac{2y^2}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2},$$

$$= -\frac{2}{\sqrt{3}} \quad \text{when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}.$$